Introduction to the Analysis of Algorithms

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Outline

- How can we measure and compare algorithms meaningfully?
- *O* notation
- Analysis of a few interesting algorithms

Introduction

- **How do we measure and compare algorithms meaningfully** given that the same algorithm will run at different speeds and will require different amounts of space when run on different computers or when implemented in different programming languages?

- **Example**: let us consider a sorting algorithm for sorting an array `A[0:n-1]`.

Selection Sorting Algorithm

```
void SelectionSort(InputArray A)
{
    int MinPosition, temp, i, j;

    for (i=n-1; i>0; --i){
        MinPosition=i;
        for (j=0; j<i; ++i){
           if (A[j] < A[MinPosition]){
             MinPosition=j;
           }
         }
         temp=A[i];
         A[i]=A[MinPosition];
         A[MinPosition]=temp;
    }
}
```

Comments

- The previous algorithm sorts the elements of an input array `A` in **descending order** using the technique of **selection sorting**.
- The algorithm proceeds in steps controlled by the outer `for` statement. In each such step the `i`-th element of the array is interchanged with the minimum element among elements `0` to `i-1`.
- The minimum element is computed by the inner for statement.
- The interchanging steps start with the last element of the array and proceed down to the first element.

Running Times in Seconds to Sort an Array of 2000 Integers (around the Year 1995 ☺)

Type of Computer	Time
Computer A	51.915
Computer B	11.508
Computer C	2.382
Computer D	0.431
Computer E	0.087

- In C we can use a library function like `clock()` from `time.h` to measure the CPU time it takes to execute some part of our program (see https://stackoverflow.com/questions/459691/best-timing-method-in-c?noredirect=1&lq=1 for discussion).
- Computers A, B, etc. up to E are progressively faster.
- The algorithm runs faster on faster computers.

More Measurements

- In addition to trying different computers, we could try **different programming languages** and **different compilers**.
- Shall we take all these measurements to decide whether an algorithm is better than another one?

A More Meaningful Criterion

- We can observe that algorithms usually **consume resources (e.g., time and space)** in some fashion that depends on the **size of the problem** solved.
- Usually, the bigger the size of a problem, the more resources an algorithm consumes.
- We usually use $\boldsymbol{n}$ to denote the size of the problem.
- **Examples of sizes**: the length of a list that is searched, the number of items in an array that is sorted etc.

`SelectionSort` Running Times in Milliseconds on Two Types of Computers (around the Year 1995 again)

Array Size n	Computer 1	Computer 2
125	12.5	2.8
250	49.3	11.0
500	195.8	43.4
1000	780.3	172.9
2000	3114.9	690.5

Two Curves Fitting the Previous Data

- If we plot these numbers on a graph and try to fit curves to them, we find that they lie on the following two curves:
  $f_1(n) = 0.0007772n^2 + 0.00305n + 0.001$
  $f_2(n) = 0.0001724n^2 + 0.00040n + 0.100$

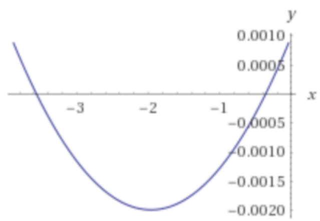
$f_1(x)$

Input:

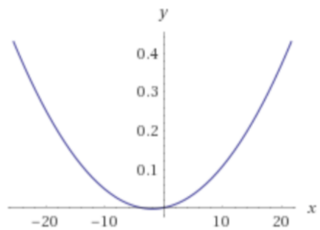
$0.0007772\,x^2 + 0.00305\,x + 0.001$

Open code

Plots:



(x from −3.9 to 0)



(x from −25.5 to 21.6)

Geometric figure:

Properties

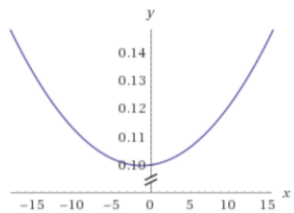
$f_2(x)$

Input:

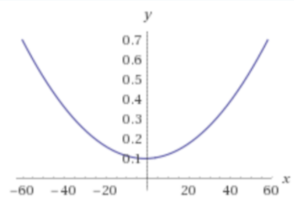
$0.0001724\,x^2 + 0.0004\,x + 0.1$

Open code

Plots:



(x from −17.9 to 15.6)



(x from −60.2 to 57.8)

Geometric figure:

Properties

parabola

Discussion

- The curves on the previous slide have the **quadratic** form $f(n) = an^2 + bn + c$.
- The difference between the two curves is that they have **different constants** a, b and c.
- Even if we implement `SelectionSort` on another computer using another programming language and another compiler, the curve that we will get will be of the same form.
- So, even though the particular measurements will change under different circumstances, **the shape of the curve** will remain the same.

Complexity Classes

- The running times of various algorithms belong to different **complexity classes**.
- Each complexity class is characterized by a **different family of curves**.
- All of the curves in a given complexity class share **the same basic shape**. The shape is characterized by an equation that gives running times as a function of problem size.

O-notation

- This notation is used in Computer Science for taking about the time complexity of an algorithm.
- For `SelectionSort`, the time complexity is $O(n^2)$.
- We find this complexity by taking the **dominant term** an^2 of the expression $an^2 + bn + c$ and throwing away the constant coefficient a.

O-notation (cont'd)

- Let us consider the equation $f(n) = an^2 + bn + c$ with $a = 0.0001724$, $b = 0.0004$ and $c = 0.1$. Then we have the following table:

n	$f(n)$	an^2	n^2 term as % of total
125	2.8	2.7	94.7
250	11.0	10.8	98.2
500	43.4	43.1	99.3
1000	172.9	172.4	99.7
2000	690.5	689.6	99.9

O-notation (cont'd)

- We conclude that **the lesser term** $bn + c$ **contributes very little** to the value of $f(n)$ even though c is 250 times more than b and b is more than two times a. Thus, we can **ignore this lesser term.**
- We will also **ignore the constant of proportionality** a in an^2 since we want to concentrate in the general shape of the curve. $\boldsymbol{a}$ **will differ for different implementations on different computers.**

Some Common Complexity Classes

O-notation	Adjective Name
$O(1)$	Constant
$O(\log n)$	Logarithmic
$O(n)$	Linear
$O(n \log n)$	n log n
$O(n^2)$	Quadratic
$O(n^3)$	Cubic
$O(2^n)$	Exponential
$O(10^n)$	Exponential
$O(2^{2^n})$	Doubly exponential

Comparing Complexity Classes

- Let us assume that we have an algorithm A that runs on a computer that executes one step of this algorithm every microsecond.
- Let us assume that $f(n)$ is the number of steps required by A to solve a problem of size n.
- Then we have the following table.

Running Times for Algorithm A

$f(n)$	$n = 2$	$n = 16$	$n = 256$	$n = 1024$
1	1 µsec	1 µsec	1 µsec	1 µsec
$log_2 n$	1 µsec	4 µsec	8 µsec	10 µsec
n	2 µsec	16 µsec	256 µsec	1.02 ms
$n\ log_2 n$	2 µsec	64 µsec	2.05 ms	10.2 ms
n^2	4 µsec	25.6 µsec	65.5 ms	1.05 secs
n^3	8 µsec	4.1 ms	16.8 ms	17.9 min
2^n	4 µsec	65.5 ms	3.7 x 10^{63} years	3.7 x 10^{294} years

Size of Largest Problem Algorithm A Can Solve in $Time \leq T$

Number of steps is	T = 1 min	T = 1hr
n	6 x 10^7	3.6 x 10^9
$n\ log_2 n$	2.8 x 10^6	1.3 x 10^8
n^2	7.75 x 10^3	6.0 x 10^4
n^3	3.91 x 10^2	1.53 x 10^3
2^n	25	31
10^n	7	9

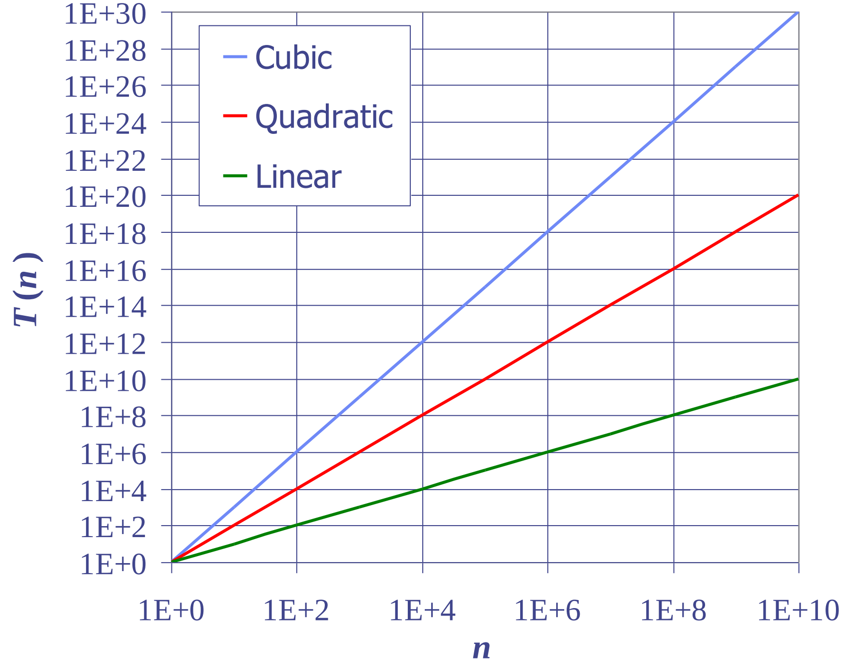
Time Complexity Discussion

- **1**: This is the case when all the statements in a program are executed a **constant** number of times.
- $\mathbf{log}\,\boldsymbol{n}$: When the time complexity of an algorithm is **logarithmic**, the algorithm runs a little bit slower when n increases. This time complexity is found in algorithms that solve a problem by transforming it into a series of smaller problems, reducing in each step the size of the problem by a constant amount. Every time n doubles, $\log n$ increases only by a constant.
- $\boldsymbol{n}$: When the time complexity of an algorithm is **linear** what happens usually is that a small part of the processing takes place for each element of the input. When n doubles, the running time of the algorithm doubles too. This time complexity is optimal for an algorithm that needs to process n inputs or to output n outputs.

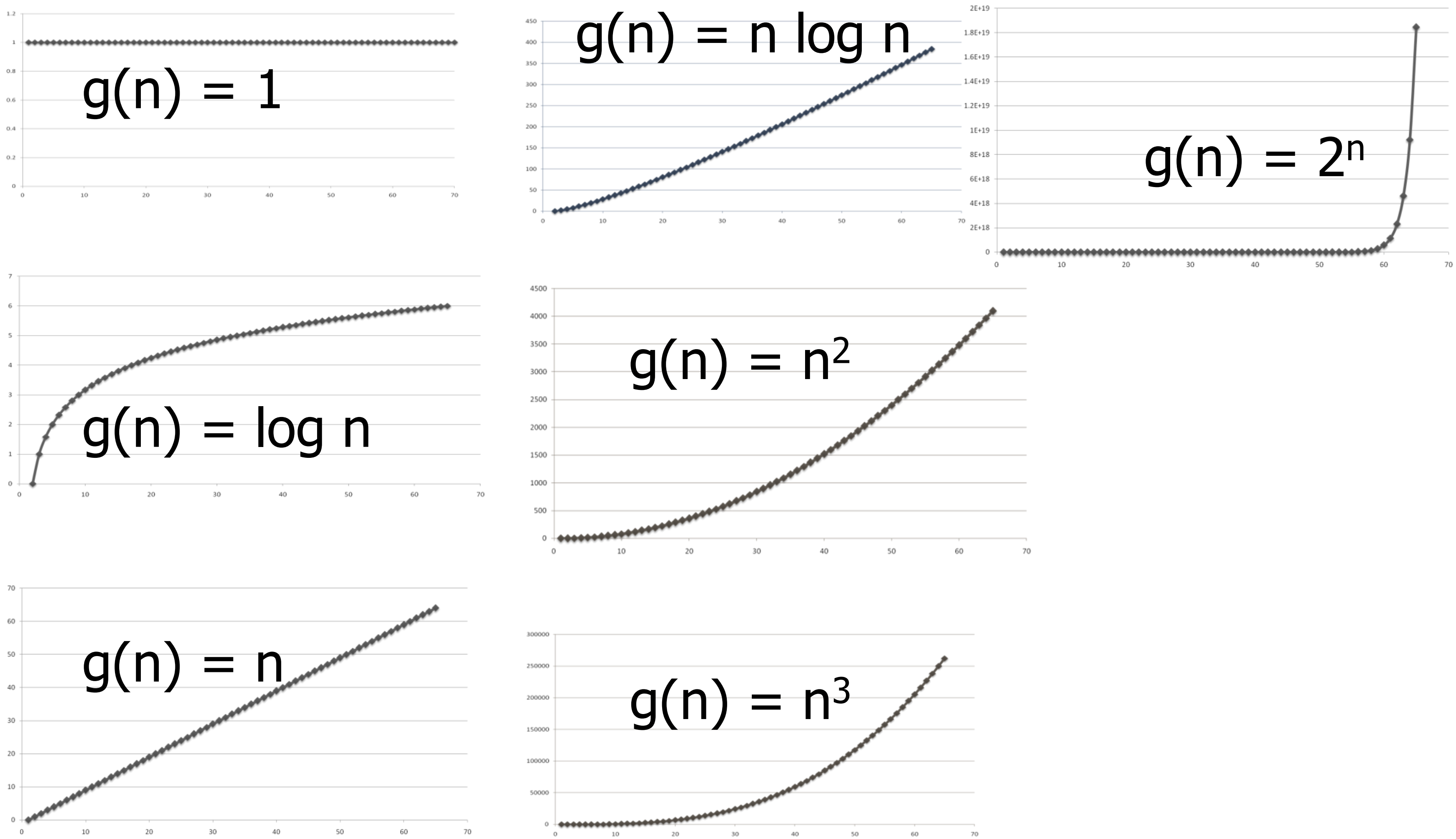
Time Complexity Discussion (cont'd)

- $\boldsymbol{n \log n}$: This time complexity appears when an algorithm solves a problem by dividing it into smaller problems and combining the partial solutions. When n doubles, the time complexity more than doubles (but it is not very far from the double).
- $\boldsymbol{n^2}$: When the time complexity is **quadratic**, the algorithm is practically useful only for small problems. Quadratic running times usually appear in algorithms that process pairs of elements of a problem (e.g., with two nested loops). When n doubles, the running time increases four times.
- $\boldsymbol{n^3}$: Similarly, an algorithm that processes triples of elements of a problem (e.g., usually with three nested loops) has **cubic** running time. It is useful only for small problem sizes. When n doubles, the running time increases eight times.
- $\boldsymbol{2^n}$: When the time complexity of an algorithms is **exponential**, the algorithm can be used in practice only for very small problem sizes. This is usually the case with algorithms that solve a problem by a brute-force method. When n doubles, the running time of the algorithm becomes the square of the previous time.

Linear vs. Quadratic vs. Cubic Time Complexity (Logarithmic Scale Plot)

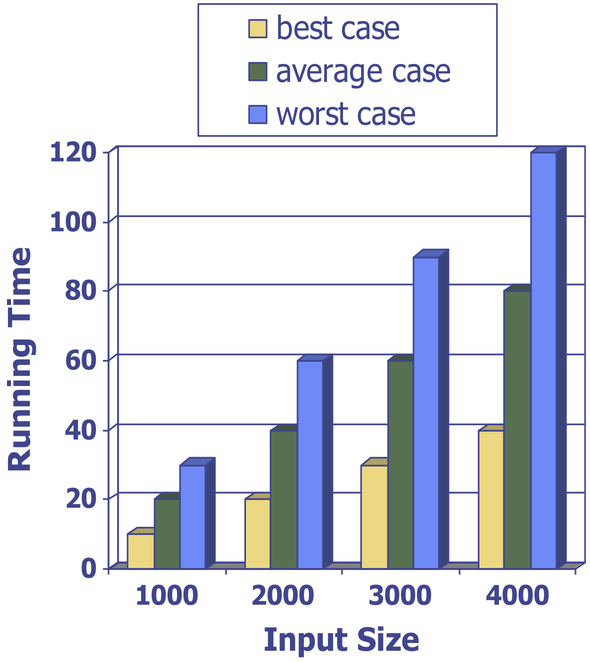


Seven Complexity Functions



Time Complexity Cases

- We may consider the following cases:
  - Worst case
  - Best case
  - Average case



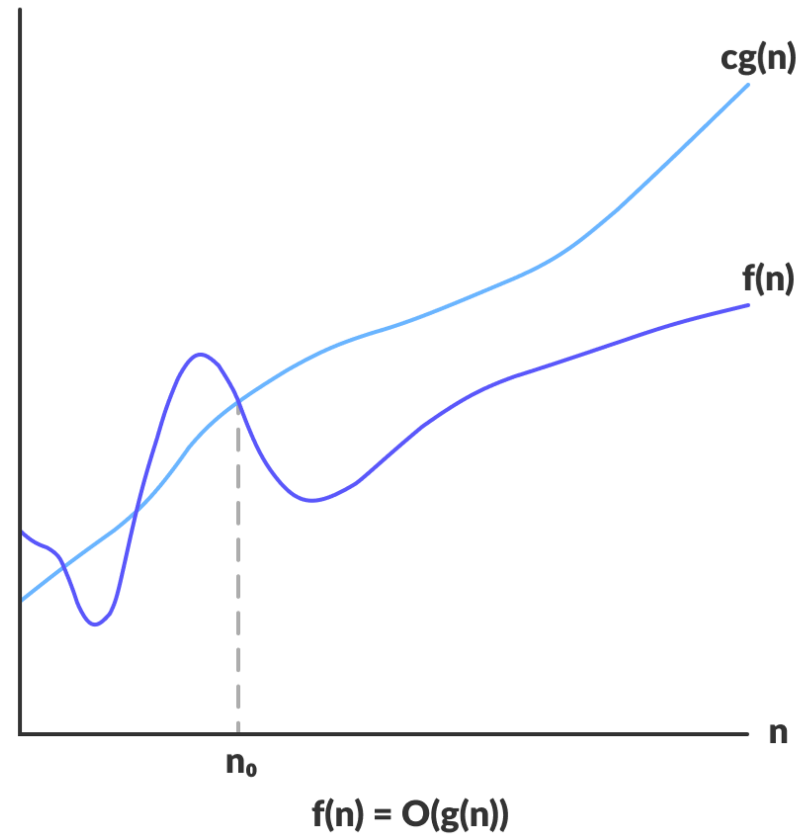
Comments

- We usually study **worst case time complexity** because it is easier than average case and more important for applications.

Formal Definition of O-notation

- We say that $\boldsymbol{f}(\boldsymbol{n})$ **is** $\boldsymbol{O}(\boldsymbol{g}(\boldsymbol{n}))$ if there exist two positive constants K and n_0 such that $|f(n)| \leq K|g(n)|$ for all $n \geq n_0$.

Graphically (with K=c)



Three Ways of Saying it in Words

- Let us assume that f and g are positive functions. Then:
  - $f(n)$ is $O(g(n))$ provided the curve $K \times g(n)$ can be made to lie above the curve for $f(n)$ whenever we are to the right of some big enough value of n_0.
  - $f(n)$ is $O(g(n))$ if there is some way to choose a constant of proportionality K so that the curve for $f(n)$ is bounded above by the curve for $K \times g(n)$ whenever n is big enough (i.e., when $n \geq n_0$).
  - $f(n)$ is $O(g(n))$ if for all but finitely many small values of n, the curve for $f(n)$ lies below the curve for some suitably large constant multiple of $g(n)$.

Example of a Formal Proof

- Let us suppose that a sorting algorithm A sorts a sequence of n numbers in ascending order with number of steps
$$f(n) = 3 + 6 + 9 + \cdots + 3n.$$
- We will show that the algorithm runs in $O(n^2)$ steps.
- **Proof:** We will first find a closed form for $f(n)$.

Proof (cont'd)

- Note that $f(n) = 3 + 6 + 9 + \cdots + 3n = 3(1 + 2 + \cdots + n) = 3\frac{n(n+1)}{2}$.
- Then, choosing $K = 3, n_0 = 1$ and $g(n) = n^2$, we can show that for all $n \geq 1$, the following inequality holds:

$$\frac{3n(n+1)}{2} \leq 3n^2$$

Proof (cont'd)

- Multiplying both sides of the above inequality with $\frac{2}{3}$ gives $n^2 + n \leq 2n^2$.
- Subtracting n^2 from both sides gives $n \leq n^2$.
- Dividing this inequality by n gives $n \geq 1$.
- The proof is now complete.

Practical Shortcuts for Manipulating O-notation

- In practice we can deal with O-notation in an easier way by separating the expression for $f(n)$ into a **dominant** term and **lesser** terms and throwing away the lesser terms.
- In other words: $O\left(f(n)\right) = O(dominant\ term \pm lesser\ terms) = O(dominant\ term)$

Scale of Strength for O-notation

- We can rank the usual complexity functions on the following **scale of strength** so it is easy to determine the dominant term and the lesser terms:

$$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n) < O(10^n)$$

Example

- $O\left(6n^3 - 15n^2 + 3n\log n\right) = \mathrm{O}\left(6n^3\right) = O(n^3)$
- Let us see why we are allowed to do the above. Notice that we have $6n^3 - 15n^2 + 3n\log n < 6n^3 + 3n\log n < 6n^3 + 3n^3 < 9\,n^3$
- This is the inequality that the definition of O-notation needs for $K = 9$ and $n \geq 1$.

Ignoring Bases of Logarithms

- When we use O-notation, we can **ignore the bases of logarithms** and assume that all logarithms are in base 2.
- Changing the bases of logarithms involves **multiplying by constants**, and constants of proportionality are ignored by O-notation.
- For example, $\log_{10} n = \frac{\log_2 n}{\log_2 10}$. Notice now that $\frac{1}{\log_2 10}$ is a constant.

$O(1)$

- It is easy to see why the $O(1)$ notation is the right one for constant time complexity.
- Suppose that we can prove that an algorithm A runs in a number of steps $f(n)$ that are always less than K steps for all n. Then $f(n) \leq K \times 1$ for all all $n \geq 1$. Therefore $f(n)$ is $O(1)$.

Some Algorithms and Their Complexity

- Sequential search
- Binary search
- Selection sort
- Recursive selection sort
- Towers of Hanoi

Analysis of Sequential Searching

- Suppose we have an array $A[0:n-1]$ that contains distinct keys K_i $(1 \leq i \leq n)$ and assume that K_i is stored in position $A[i-1]$.

- **Problem**: we are given a key K and we would like to determine its position in $A[0:n-1]$.

An Algorithm for Sequential Searching

```
#define n 100
typedef int Key;
typedef Key SearchArray[n];

int SequentialSearch(Key K, SearchArray A)
{
    int i;

    for (i=0; i<n; ++i){
        if (K==A[i]) return i;
    }
    return(-1);
}
```

Complexity Analysis

- The amount of work done to locate key K **depends on its position** in $A[0{:}\, n-1]$.
- For example, if K is in $A[0]$, then we need only one comparison.
- In general, if K is in $A[i-1]$, then we need i comparisons.

Complexity Analysis (cont'd)

- **Best case**: This is when K is in $A[0]$. The complexity is $O(1)$.
- **Worst case**: This is when K is in $A[n-1]$. The amount of work is $an+b$ where a and b are constants. Therefore the complexity is $O(n)$.
- **Average case**: Let us assume that each key is equally likely to be used in a search. The average can then be computed by taking the total of all the work done for finding all the different keys and dividing by n.

Complexity Analysis (cont'd)

- The work needed to find the i-th key K_i is of the form $a\ i + b$ for some constants a and b. Therefore:

$$Total = \sum_{i=1}^{n}(ai + b) = a\sum_{i=1}^{n} i + \sum_{i=1}^{n} b =$$
$$= a\frac{n(n+1)}{2} + bn$$

- Now the average is:

$$Average = \frac{Total}{n} = a\frac{n+1}{2} + b = \frac{a}{2}n + \left(\frac{a}{2} + b\right)$$

- Therefore, the average is $O(n)$.

Binary Search

- For sorted arrays, we have an algorithm more efficient than sequential search: **binary search**.

Binary Search (cont'd)

- The problem to be addressed in binary searching is to find the position of a **search key** K in an ordered array A[0:n-1] of distinct keys arranged in ascending order:

  A[0] < A[1] < … < A[n-1].

- The algorithm chooses the key in the middle of A[0:n-1], which is located at A[Middle], where Middle=(0+(n-1))/2, and compares the search key K and A[Middle].
- If K==A[Middle], the search terminates successfully.
- If K < A[Middle] then further search is conducted among the keys to the left of A[Middle].
- If K > A[Middle] then further search is conducted among the keys to the right of A[Middle].

Iterative Binary Search

```
int BinarySearch(Key K)
{
    int L, R, Midpoint;

    /* Initializations */
    L=0;
    R=n-1;

    /* While the interval L:R is  non-empty, test key K against the middle key */
    while (L<=R){
      Midpoint=(L+R)/2;
      if (K==A[Midpoint]){
        return Midpoint;
      } else if (K > Midpoint) {
        L=Midpoint+1;
      } else {
        R=Midpoint-1;
      }
    }
    /* If the search interval became empty, key K was not found */
    return -1;
}
```

Recursive Binary Search

```
int BinarySearch (Key K, int L, int R)
{
   /* To find the position of the search key K in the subarray
      A[L:R]. Note: To search for K in A[0:n-1], the initial call
      is BinarySearch(K, 0, n-1) */

   int Midpoint;

   Midpoint=(L+R)/2;

   if (L>R){
     return -1;
   } else if (K==A[Midpoint]){
     return Midpoint;
   } else if (K > A[Midpoint]){
     return BinarySearch(K, Midpoint+1, R);
   } else {
     return BinarySearch(K, L, Midpoint-1);
   }
}
```

Complexity

- Let us compute the running time of recursive binary search.
- We call an entry of our array a **candidate** if, at the current stage of the algorithm, we cannot rule out that this entry has key equal to `K`.
- We observe that a constant amount of primitive operations are executed at each recursive call of function `BinarySearch`.
- Hence the running time is proportional to the number of recursive calls performed.
- Moreover, the number of remaining candidates is reduced by **at least half** with each recursive call.

Complexity (cont'd)

- Initially, the number of candidate entries is n. After the first call to `BinarySearch`, it is at most $\frac{n}{2}$. After the second call, it is at most $\frac{n}{4}$ and so on.
- In general, after the i-th call to `BinarySearch`, the number of candidate entries is at most $\frac{n}{2^i}$.
- In the worst case (unsuccessful search), the recursive calls stop when there are no more candidate entries. Hence, the maximum number of recursive calls performed, is the smallest integer m such that $\frac{n}{2^m} < 1$.

Complexity (cont’d)

- Equivalently, $2^m > n$.
- Taking logarithms in base 2, we have $m > \log n$.
- Thus, we have $m = \lfloor \log n \rfloor$+1 which implies that the complexity of recursive `BinarySearch` is $O(\log n)$.
- The complexity of iterative `BinarySearch` is also $O(\log n)$.

Selection Sorting Algorithm

```
void SelectionSort(InputArray A)
{
    int MinPosition, temp, i, j;

    for (i=n-1; i>0; --i){
        MinPosition=i;
        for (j=0; j<i; ++i){
           if (A[j] < A[MinPosition]){
             MinPosition=j;
           }
         }
         temp=A[i];
         A[i]=A[MinPosition];
         A[MinPosition]=temp;
    }
}
```

Complexity Analysis of SelectionSort

- We start from the inner `for` statement. The `if` statement inside the `for` takes a constant amount of time a. Thus, the `for` statement takes ia time units.
- Let us now consider the outer `for`. The statements inside this `for`, except the inner `for`, take a constant amount of time b. Thus all the statements inside the outer `for` take time $ai + b$.

Complexity Analysis (cont'd)

- The outer `for` takes time

$$\sum_{i=1}^{n-1}(ai+b) = a\sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} b =$$

$$= a\frac{(n-1)n}{2} + (n-1)b =$$

$$= \frac{a}{2}n^2 + \left(b - \frac{a}{2}\right)n - b$$

- Therefore, the time complexity of the algorithm is $O\left(n^2\right)$.

Recursive SelectionSort

```
/* FindMin is an auxiliary function used by the Selection sort below */
int FindMin(InputArray A, int n)
{
    int i,j=n;
    for (i=0; i<n; ++i) if (A[i]<A[j]) j=i;
    return j;
}

void SelectionSort(InputArray A, int n)
{
    int MinPosition, temp;

    if (n>0){
       MinPosition=FindMin(A,n);
       temp=A[n]; A[n]=A[MinPosition]; A[MinPosition]=temp;
       SelectionSort(A, n-1)
    }
}
```

Analysis of Recursive SelectionSort

- To use this recursive version of `SelectionSort` to perform selection sorting on the array `A[0:n-1]`, we make the function call `SelectionSort(A,n-1)`.
- The first thing we need to do is to analyze the running time of function `FindMin` which finds the position of the smallest element in the array `A[0:n]`.
- It is easy to see that the time for this function is $an + b_1$ for suitable constants a and b_1.

Analysis of Recursive SelectionSort (cont'd)

- We now analyze the running time of recursive function `SelectionSort`.
- Let $T(n)$ stand for the cost, in time units, of calling `SelectionSort` on `A[0:n]`.
- Then the costs in `SelectionSort` are as follows:

```
if (n>0){
        Cost  an + b1
        Cost  b2
        Cost  T(n − 1)
}
```

Analysis of Recursive SelectionSort (cont'd)

- If $b = b_1 + b_2$ then the following **recurrence relation** holds for $n > 0$:
$$T(n) = an + b + T(n-1)$$
- The base case of this recurrence relation is $T(0) = c$ where c is the cost of executing `SelectionSort(A,0)`.
- To solve such recurrence relations, we can use a method called **unrolling**.

Analysis of Recursive SelectionSort (cont'd)

$$T(n) = an + b + T(n-1)$$
$$T(n) = an + b + a(n-1) + b + T(n-2)$$
$$T(n) = an + b + a(n-1) + b + a(n-2) + b + T(n-3)$$

…

$$T(n) = an + b + a(n-1) + b + a(n-2) + b + \cdots + a * 1 + b + T(0)$$

$$T(n) = an + b + a(n-1) + b + a(n-2) + b + \cdots + a * 1 + b + c$$

Analysis of Recursive SelectionSort (cont'd)

- Rearranging some of the terms so that all those with coefficients a and b are collected together, we have:

$$T(n) = (an + a(n-1) + a(n-2) + \cdots + a) + nb + c$$

$$T(n) = \sum_{i=1}^{n}(ai) + nb + c$$

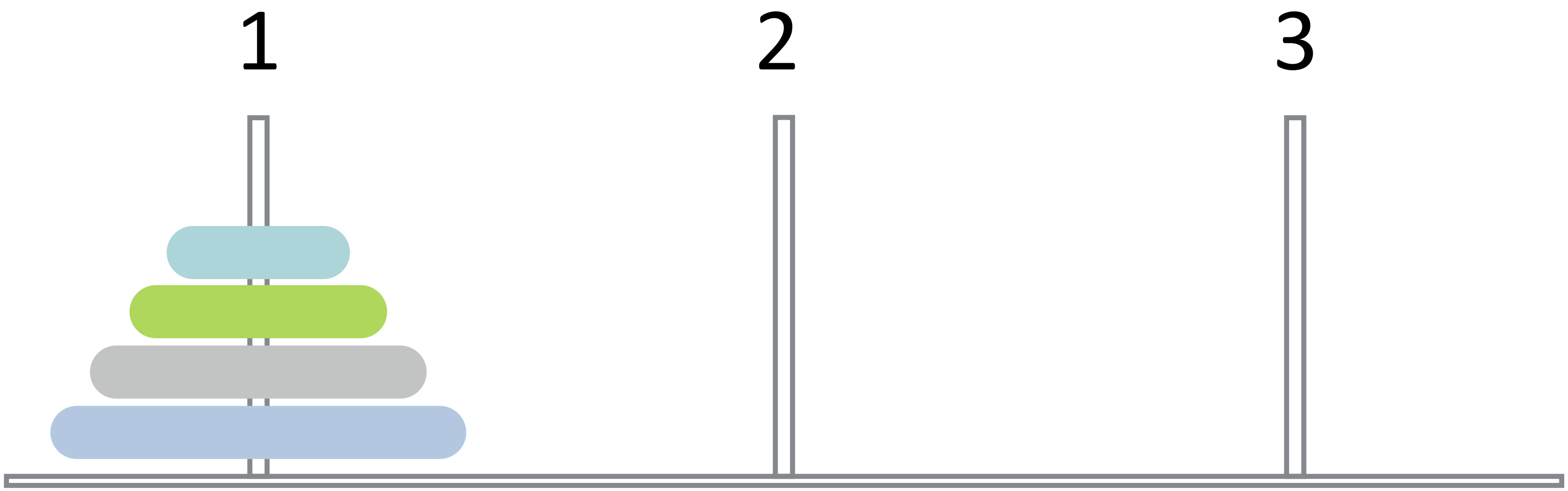
$$= a\frac{n(n+1)}{2} + nb + c$$

$$= \frac{a}{2}n^2 + \left(\frac{a}{2} + b\right)n + c$$

Analysis of Recursive SelectionSort (cont'd)

- Therefore $T(n)$ but also $T(n-1)$ is $O\left(n^2\right)$.

The Towers of Hanoi



A Recursive Solution to the Towers of Hanoi

```
void MoveTowers(int n, int start, int finish, int spare)
{
   if (n==1){
      printf("Move a disk from peg %1d to peg %1d\n", start, 
finish);
   } else {
      MoveTowers(n-1, start, spare, finish);
      printf("Move a disk from peg %1d to peg %1d\n", start, 
finish);
      MoveTowers(n-1, spare, finish, start);
   }
}
```

Analysis of Towers of Hanoi

- Let n be the number of disks to be moved. Then the running time $T(n)$ of the algorithm is given by the following recurrence relations:

$$T(1) = a$$
$$T(n) = b + 2T(n-1)$$

- We will solve these recurrence relations using the technique of **unrolling plus summation**.

Analysis of Towers of Hanoi (cont’d)

$$T(n) = b + 2T(n-1)$$
$$T(n) = b + 2\big(b + 2T(n-2)\big)$$
$$T(n) = b + 2b + 2^2T(n-2)$$
$$T(n) = b + 2b + 2^2\big(b + 2T(n-3)\big)$$
$$T(n) = b + 2b + 2^2b + 2^3T(n-3)$$
$$\dots$$
$$T(n) = b + 2b + 2^2b + \cdots + 2^{(i-1)}b + 2^iT(n-i)$$

Analysis of Towers of Hanoi (cont'd)

- When $i = n - 1$, we have:

$$T(n-i) = T\big(n-(n-1)\big) = T(n-n+1) = T(1) = a$$

- Therefore, $T(n)$ can be expressed as follows:

$$T(n) = 2^0 b + 2^1 b + 2^2 b + \cdots + 2^{(n-2)} b + 2^{(n-1)} a =$$

$$= \sum_{i=0}^{n-2} 2^i b + 2^{(n-1)} a =$$

$$= b \sum_{i=0}^{n-2} 2^i + 2^{(n-1)} a$$

Analysis of Towers of Hanoi (cont'd)

- Now we can see that the sum is a standard **geometric progression**. So we will use the fact that $\sum_{k=0}^{m} x^k = \frac{x^{m+1}-1}{x-1}$ to conclude the following:

$$\sum_{i=0}^{n-2} 2^i = \frac{2^{(n-1)} - 1}{2 - 1} = 2^{(n-1)} - 1$$

Analysis of Towers of Hanoi (cont'd)

- Therefore:

$$T(n) = b\left(2^{(n-1)} - 1\right) + 2^{(n-1)}a$$

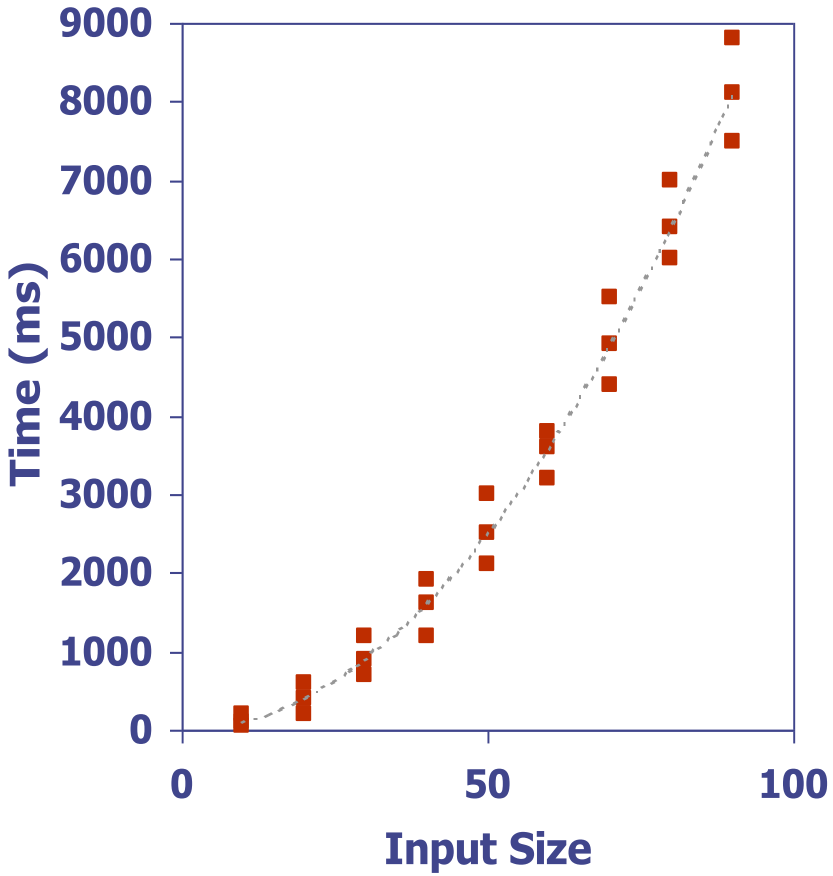
$$= (a+b)2^{(n-1)} - b = \frac{a+b}{2}2^n - b$$

- Finally, we can see that $T(n)$ is $O(2^n)$.

What O-notation Does Not Tell You

- O-notation **does not apply to small problem sizes** because in this case the constants might dominate the other terms.
- One can use **experimental testing** to select the best algorithm in this case.
- Experimental testing is also useful if we want to compare algorithms that are **in the same complexity class.**

Experimental Testing of Algorithms



Space Complexity

- In a similar way, we can measure the **space complexity** of an algorithm.

Other notations

- There also other complexity notations such as $o(n), \Theta(n), \Omega(n), \omega(n)$.
- More details in the Algorithms course.

Readings

- T. A. Standish. *Data Structures, Algorithms and Software Principles in C*.
  - Sections 5.6 and Chapter 6
- Robert Sedgewick. Αλγόριθμοι σε C.
  - Κεφ. 2
- M.T. Goodrich, R. Tamassia and D. Mount. *Data Structures and Algorithms in C++*. 2nd edition.
  - Section 9.3